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Eigenvalue problems of beams with wedge-shaped Vlasov foundations

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Abstract

Eigenvalue problems of the Euler–Bernoulli beams, fixed on an elastic foundation layer of Vlasov type with variable depth, have been investigated. The mixed finite element method based on the weak formulation is used. Displacements and moments are the primary variables of the presented mixed method. Their first derivatives are the secondary variables. The behavior matrices of the beam element have been obtained by the use of the weak formulation satisfying equilibrium and compatibility equations.

1. Introduction

Research for representation of soil by a simple mathematical model in soil–structure interaction problems, can be separated into two groups, namely Winklerian and continuous models [1]. The contact pressure distribution on the soil–beam interface, which depends on the foundation behavior (viz., rigid or flexible extreme situations) and the nature of the soil deposit, are important parameters. The subgrade modulus may be obtained from alternative approaches such as the plate load test [2–4], the consolidation test [5,6], the three-axial test [7,8], and the CBR test [6,9–11]. The basic limitation of the Winkler hypothesis lies in the fact that this model does not include any term for the dispersion of stresses over the influenced area. More accurate soil models, formally, can be obtained by means of the Favre method [12], retaining the first of two terms of the Taylor series expansion, or, more generally can be classified as the grade 2,3 of the Bharatha–Levinson series [13].

The Vlasov foundation model has been developed using continuum idealization and the variational principle [14,15].

Response functions of the Vlasov model, obtained using the principles of virtual work and reported in the literature, depend on two different parameters, k and t. The characteristic k determines the compressive strain in the elastic foundation, so it is similar to the foundation modulus. The characteristic t determines the shear strain in the elastic foundation, so it defines the load-spreading capacity of the foundation. In this model, the modulus of elasticity can be assumed to be varying all over the depth.

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In many geotechnical applications, the soil can be considered as an elastic foundation whose thickness H varies linearly in the x-direction, like a wedge.

Recently, some biparametric models for an elastic foundation have been proposed for foundation in the shape of a wedge, by Marzeda et al. [16]. Their modelling procedure starts from the linear elasticity equations into which they introduce some simplifying assumptions based on concepts of decay functions. Their simplified models are described by ordinary differential equations. In their study, the interaction between the rigid plate and the biparametric wedge foundation is considered.

Following the Vlasov theory given in Ref. [15], eigenvalue problems of the Euler–Bernoulli beams on a Vlasov foundation in the shape of a wedge have been investigated by employing the mixed finite element method.

In the mixed finite element method, the primary variables of the beam element are the nodal values of the displacements and the moments, which are both basic parameters in the analysis and design of structures. This is the main advantage of the mixed method in comparison with classical one-field finite element models. Furthermore, simple linear interpolation functions may be used for displacement and moment fields, because the C^0 continuous functions are sufficient in the weak formulation.

In this study, behavior matrices of the mixed finite element are derived by using linear and cubic interpolation functions for the moment and displacement fields, respectively. A similar mixed finite element method has been proposed by Ergüven and Gedikli for Timoshenko beams on a Winkler foundation, [17].

Improvement of the mixed method proposed in Ref. [17] and the analysis of elastic beam on an elastic wedge foundation of Vlasov, which is rare in the literature, are the main objectives of this study.

The sensitivity of the method presented to the number of beam elements has been examined. The effects of the parameters, which are the bending rigidity of the beam, the slope of the rigid bed, and the depth of the foundation layer, on the natural frequencies and static buckling loads of the system have been investigated. In addition, free vibration of a pre-stressed beam fixed on to the elastic wedge foundation has also been considered.

2. Elastic wedge foundation

An elastic wedge foundation and a beam resting on it are illustrated in Fig. 1. It is known from the practice that the vertical displacement plays the main role and the horizontal displacements in the foundation can be neglected, i.e., $u(x, y) = 0$, [15].

The vertical displacement of a point, (x, y), of the foundation can be expressed as

$$v(x, y) = V(x)\phi(x, y). \tag{1}$$

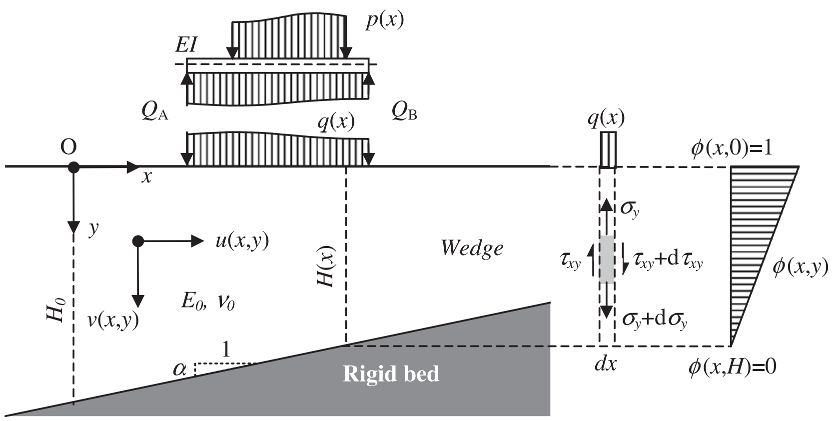

Fig. 1. Elastic wedge foundation and beam resting on it.

Consider a wedge as in Fig. 1, whose depth H varies linearly in the x-direction as

$$H(x) = H_0 - \alpha x, \tag{2}$$

where H_0 is the depth at $x = 0$, and α is the tangent of the slope angle of the rigid bed.

The behavior of deep foundation layers can be closely modelled by the following shape function [15]:

$$\phi_d(x,y) = \frac{\sinh \gamma(H(x) - y)}{\sinh \gamma H(x)}, \tag{3}$$

where γ is a constant, which determines the rate of decrease of displacements with depth.

Since σ_y can be assumed to remain constant throughout a vertical strip for a shallow foundation, shape function for shallow foundation can be defined as a linear function of y as follows:

$$\phi(x,y) = \frac{H(x) - y}{H(x)}. \tag{4}$$

If ϕ_d is expanded to a Taylor series in terms of γ in the neighborhood of zero, the first term is exactly equal to ϕ.

According to Eqs. (1), (2), (4) and the constitutive equations of linear elasticity, the normal and shear stresses can be obtained as

$$\sigma_y = -\frac{E_s}{1 - v_s^2}\frac{V}{H}, \quad \tau_{xy} = \frac{3k_1}{b}\left(\phi\, V_{,x} - \alpha\frac{y}{H^2}V\right),$$
$$k_1 = \frac{E_s b}{6(1 + v_s)} \tag{5}$$

where E_s, v_s and b represent the elasticity modulus, Poisson ratio and the thickness of the wedge layer, respectively.

Let a field of virtual displacement be

$$\bar{v}(x,y) = \bar{V}(x)\phi(x,y). \tag{6}$$

The internal and external forces on the infinitesimal element given in Fig. 1 are $b\,\mathrm{d}\sigma_y\,\mathrm{d}x$ and $b\,\mathrm{d}\tau_{xy}\,\mathrm{d}y$, respectively. The total work of these forces on the virtual displacement of $\bar{v}$ is equal to zero. Therefore,

$$b(\mathrm{d}\tau_{xy}\,\mathrm{d}y + \mathrm{d}\sigma_y\,\mathrm{d}x)\bar{v} = 0. \tag{7}$$

Rearranging this expression, the total work for the elementary strip given in Fig. 1 can be obtained as follows:

$$b\,\mathrm{d}x\int_0^{H(x)} (\tau_{xy,x}\bar{v} + \sigma_{y,y}\bar{v})\,\mathrm{d}y = 0. \tag{8}$$

Integrating the second term by parts and omitting the term of $\mathrm{d}x$, Eq. (8) leads

$$b\int_0^{H(x)} \left(\tau_{xy,x}\bar{v} - \sigma_y\bar{v}_{,y}\right)\mathrm{d}y + b\,\sigma_y\bar{v}\Big|_0^{H(x)} = 0. \tag{9}$$

Substituting Eq. (6) and the boundary conditions, which are $\bar{v}(x,H) = 0$, $\bar{v}(x,0) = \bar{V}(x)$ and $b\sigma_y(x,0) = -q(x)$, into Eq. (9),

$$b\int_0^{H(x)} (\tau_{xy,x}\phi - \sigma_y\phi_{,y})\,\mathrm{d}y + q(x) = 0 \tag{10}$$

is obtained, where $q(x)$ is the distributed load on x-axis. Substituting Eq. (4) into Eq. (5) and then into Eq. (10), the Eulerian differential equation can be written as [15]

$$-k_1 H V_{,xx} + k_2 V_{,x} + kV/H = q, \tag{11}$$

where $k_2 = k_1\alpha$, $k = k_1 r^2$ and $r^2 = \alpha^2 + 6/(1 - v_s)$.

It is known that, for the plane strain problem, elastic constants are given as follows:

$$E_0 = \frac{E_s}{1 - \nu_s^2}, \quad \nu_0 = \frac{\nu_s}{1 - \nu_s}. \tag{12}$$

Meshing an area, which is large enough so that displacements, strains and consequently stresses are negligible on its boundaries, is one of the well-known techniques of FEM in solving a problem established on a semi-infinite region. If available, the use of a closed form of semi-infinite finite element is another approach. Difficulties in numerical implementations make the closed-form solutions be attractive for regions $x > x_B$ and for $x < x_A$ as shown in Fig. 2.

These regions of the wedge can be modeled as a fictitious elastic springs, k_A and k_B, respectively. For this purpose, the homogeneous form of Eulerian differential equation given by Eq. (11), can be written for a particular case as follows:

$$-H^2 V_{,xx} + \alpha H V_{,x} + r^2 V = 0. \tag{13}$$

In order to solve this, let $V(x) = H^{\lambda}(x)$. Then, the roots λ_1 and λ_2 are obtained as

$$\lambda_1 = -\lambda, \quad \lambda_2 = \lambda, \quad \lambda = r/\alpha \tag{14}$$

and the solution of Eq. (13) is found as follows:

$$V(x) = c_1 H^{-\lambda}(x) + c_2 H^{\lambda}(x), \tag{15}$$

where c_1 and c_2 are integration constants. The displacements of free surface of elastic foundation tend to zero for the limits $x \to -\infty$ (in this case c_2 vanishes) and $x \to H_0/\alpha$ (in this case c_1 vanishes). As a result,

$$V(x) = \begin{cases} V_0 H_0^{\lambda} H^{-\lambda}(x), & x \leqslant 0, \\ V_0 H_0^{-\lambda} H^{\lambda}(x), & x \geqslant 0. \end{cases} \tag{16}$$

$V(x)$ cannot be accurately calculated by Eq. (16), when rigid bed is near to the horizontal position, i.e. $\alpha \to 0$. To achieve this problem, displacement function can be expanded into a power series in term of α in the neighborhood of zero as

$$V(x) \cong \sum_{i=0}^{N} \frac{\alpha^i}{i!} \lim_{\alpha \to 0} \frac{\partial^i V(x)}{\partial \alpha^i}. \tag{17}$$

In this series the first term corresponding to $i = 0$ is

$$\lim_{\alpha \to 0} V(x) = \begin{cases} V_0 \mathrm{e}^{rx/H_0}, & x \leqslant 0, \\ V_0 \mathrm{e}^{-rx/H_0}, & x \geqslant 0. \end{cases} \tag{18}$$

The same result can also be found writing $\alpha = 0$ in Eq. (13). It has also given by Vlasov [15]. This rule will be used for determining the behavior matrices in Section 5 as well.

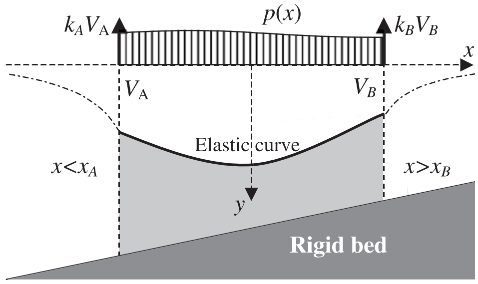

Fig. 2. Wedge foundation, elastic curve and fictitious elastic spring forces.

Following the Vlasov theory, in order to obtain coefficients of the fictitious elastic springs, a virtual displacement field can be defined as

$$\bar{v}(0,y) = \bar{V}(0)\phi(0,y). \tag{19}$$

Considering a shear stress field associated with displacement field given by Eq. (16) for the region $x \leqslant 0$, the work of the shear force S_A (called as generalized shear force by Vlasov [15]), shown in Fig. 3a, on the virtual displacement $\bar{V}(0)$ is equal to the work of shear forces $b\tau_{xy}\,\mathrm{d}y$ on the same virtual displacement $\bar{v}(0,y)$. As a result,

$$S_A\bar{V} = b\int_0^{H_0} \tau_{xy}\bar{v}\,\mathrm{d}y. \tag{20}$$

Substituting Eqs. (5) and (19) into Eq. (20), the generalized shear force S_A, and similarly S_B shown in Fig. 3b, are obtained as

$$\begin{aligned} S_A &= k_A V_0, \quad k_A = (\lambda - 1/2)k_2, \\ S_B &= -k_B V_0, \quad k_B = (\lambda + 1/2)k_2, \end{aligned} \tag{21}$$

where k_A and k_B will be called as the coefficients of the fictitious elastic springs at the points $x = 0^-$ and 0^+, Fig. 3, respectively. It is interesting that these coefficients do not depend on the depth of the foundation layer.

3. Elastic beam

In the Bernoulli–Euler beam theory, the relationships between transverse deflection $w(x)$, the rotation of cross-section plane θ, and the curvature κ, can be given as follows:

$$\theta = w_{,x}, \quad \kappa = \theta_{,x}. \tag{22}$$

In the absence of distributed moment loads, the equilibrium equations of the beam resting on an elastic foundation described above are

$$T_{,x} = -p + q, \tag{23}$$

$$M_{,x} = T. \tag{24}$$

where M, T, p and q are bending moment, shear force, distributed load on the beam and distributed soil reaction, respectively. The relationship between moment and curvature is known as

$$M = -\mathrm{EI}\,\theta_{,x}, \tag{25}$$

where EI is the bending rigidity of the beam. The mixed form of governing equations can be summarized by using Eqs. (22)–(25), as follows:

$$M_{,xx} = -p + q, \tag{26}$$

$$w_{,xx} = \theta_{,x}. \tag{27}$$

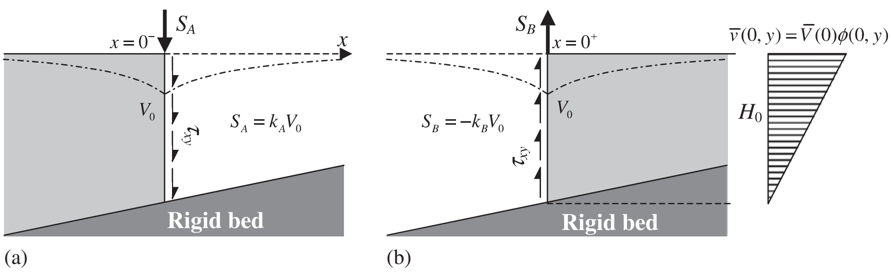

Fig. 3. Generalized shear forces: (a) at $x = 0^-$ and (b) at $x = 0^+$.

The axial compression and the inertial forces result to the transversely distributed forces on the beam as follows:

$$p = -Pw_{,xx} + m\omega^2 w, \tag{28}$$

where m, ω and P represent the mass per unit length of the beam, the natural frequency of the system and the axially compressive load, respectively.

Eq. (28) leads to the related eigenvalue problems:

(a) *Free vibration of the system*: $P = 0$ and $\omega \neq 0$.
(b) *Buckling* (*static stability*): $P \neq 0$ and $\omega = 0$.
(c) *Free vibration of the system with axially pre-stressed beam*: $P \neq 0$ and $\omega \neq 0$.

4. Formulation of beam–foundation interaction by weak form

The weak forms of Eqs. (26) and (27) over an element (x_A, x_B) are developed by using the usual procedure, Reddy [18]. In this study, the mass of the foundation layer is neglected for the sake of simplicity. It will be considered in the later studies.

It should be noticed that the deflection of the beam $w(x)$ is equal to the deflection of the foundation $V(x)$, since the beam is assumed to be fixed on the foundation. Substituting Eqs. (11) and (28) into Eq. (26) and multiplying the result by a weighting function $u(x)$, additionally, substituting Eqs. (25) into Eq. (27) and then multiplying the result by a weighting function $v(x)$ and integrating them over the element (x_A, x_B), the weighted residual forms are obtained as follows:

$$\int_{x_A}^{x_B} u(M_{,xx} - Pw_{,xx} + m\omega^2 w + k_1 H w_{,xx} - k_2 w_{,x} - kw/H)\,\mathrm{d}x = 0, \tag{29}$$

$$\int_{x_A}^{x_B} v\left(w_{,xx} + \frac{M}{\mathrm{EI}}\right)\mathrm{d}x = 0, \tag{30}$$

where subscripts A and B correspond to the left and right ends of the beam, respectively. Integrating Eqs. (29) and (30) by parts, the weighted residual forms become

$$\int_{x_A}^{x_B}\left(u_{,x}M_{,x} - Pu_{,x}w_{,x} - m\omega^2 uw + k_1 H u_{,x} w_{,x} + k\frac{uw}{H}\right)\mathrm{d}x - u(T - P\theta + k_1 H\theta)\big|_{x_A}^{x_B} = 0, \tag{31}$$

$$\int_{x_A}^{x_B}\left(v_{,x}w_{,x} - \frac{vM}{\mathrm{EI}}\right)\mathrm{d}x - v\theta\big|_{x_A}^{x_B} = 0, \tag{32}$$

where $T = M_{,x}$ and $\theta = w_{,x}$ which are the coefficients of the weighting functions in the boundary integrals.

Because of the physical meaning of these boundary terms, u must be equaled to the transverse deflection and/or its variation, while v must be equaled to the bending moment and/or its variation, i.e., $u \to w$ (or δw) and $v \to M$ (or δM). Substituting these chosen weighting functions into functionals given by Eqs. (31) and (32) and then combining them, a single variational form can be obtained as follows:

$$\delta\left\{\int_{x_A}^{x_B} w_{,x}M_{,x}\,\mathrm{d}x - \frac{P}{2}\int_{x_A}^{x_B} w_{,x}^2\,\mathrm{d}x - \frac{m\omega^2}{2}\int_{x_A}^{x_B} w^2\,\mathrm{d}x + \frac{k_1}{2}\int_{x_A}^{x_B} Hw_{,x}^2\,\mathrm{d}x \right.$$
$$\left. + \frac{k}{2}\int_{x_A}^{x_B}\frac{w^2}{H}\mathrm{d}x - \frac{1}{2\mathrm{EI}}\int_{x_A}^{x_B} M^2\,\mathrm{d}x - M\theta\big|_{x_A}^{x_B} - w(T - P\theta + k_1 H\theta)\big|_{x_A}^{x_B}\right\} = 0. \tag{33}$$

The last terms in Eq. (33) equal to the external transverse loads P_A and P_B, applied to ends of the beam element. Thus, $P_A = -T_A + P\theta_A - k_1 H_A \theta_A$ and $P_B = T_B - P\theta_B + k_1 H_B \theta_B$. At the node of two adjacent elements, $P_B^{\text{first}} + P_A^{\text{second}}$ equals to the value of the load applied to this node. In this work, this kind of singular loads are not considered.

The boundary terms in Eqs. (31) and (32) indicate that the primary variables w and M are related with the essential boundary conditions, while the secondary variables θ and T are related with the natural boundary conditions. From now on, the presented FEM will be called as the mixed FEM.

5. Mixed finite element model

It is known that the basic system of the stiffness method is a beam with clamped ends. On the contrary, here the basic system of the mixed method is a selected to be a simple beam with two hinges at the ends. In both methods, the displacement function for the beam element can be represented by the sum of elastic lines depending on the nodal parameters. In this manner, functions of displacement and moment can be written as

$$\begin{aligned} w(x) &= \boldsymbol{\varphi}^{\mathrm{T}}(x)\mathbf{w} + \boldsymbol{\psi}^{\mathrm{T}}(x)\mathbf{M} \quad & \mathbf{w}^{\mathrm{T}} &= \{w_A, w_B\}, \\ M(x) &= \boldsymbol{\varphi}^{\mathrm{T}}(x)\mathbf{M} & \mathbf{M}^{\mathrm{T}} &= \{M_A, M_B\}, \end{aligned} \tag{34}$$

where $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are vectors of shape functions for the beam element. The variational calculus emphasizes that admissible displacements $w(x)$ and moments $M(x)$ must be at least C^0 continuous and satisfy exactly the any type boundary conditions, in view of the fact that Eq. (33) includes up to first derivatives in w and M.

Substituting Eqs. (34) into Eq. (33), the variational form becomes

$$\begin{aligned} \delta\Bigg\{ & \mathbf{w}^{\mathrm{T}}\int_{x_A}^{x_B} \boldsymbol{\phi}_{,x}\boldsymbol{\phi}_{,x}^{\mathrm{T}}\,\mathrm{d}x\mathbf{M} + \mathbf{M}^{\mathrm{T}}\int_{x_A}^{x_B} \boldsymbol{\psi}_{,x}\boldsymbol{\phi}_{,x}^{\mathrm{T}}\,\mathrm{d}x\mathbf{M} - \frac{P}{2}\Bigg\{\mathbf{w}^{\mathrm{T}}\int_{x_A}^{x_B} \boldsymbol{\phi}_{,x}\boldsymbol{\phi}_{,x}^{\mathrm{T}}\,\mathrm{d}x\mathbf{w} \\ & +\mathbf{w}^{\mathrm{T}}\int_{x_A}^{x_B} \boldsymbol{\phi}_{,x}\boldsymbol{\psi}_{,x}^{\mathrm{T}}\,\mathrm{d}x\mathbf{M} + \mathbf{M}^{\mathrm{T}}\int_{x_A}^{x_B} \boldsymbol{\psi}_{,x}\boldsymbol{\phi}_{,x}^{\mathrm{T}}\,\mathrm{d}x\mathbf{w} + \mathbf{M}^{\mathrm{T}}\int_{x_A}^{x_B} \boldsymbol{\psi}_{,x}\boldsymbol{\psi}_{,x}^{\mathrm{T}}\,\mathrm{d}x\mathbf{M}\Bigg\} \\ & -\frac{m\omega^2}{2}\Bigg\{\mathbf{w}^{\mathrm{T}}\int_{x_A}^{x_B} \boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\,\mathrm{d}x\mathbf{w} + \mathbf{w}^{\mathrm{T}}\int_{x_A}^{x_B} \boldsymbol{\phi}\boldsymbol{\psi}^{\mathrm{T}}\,\mathrm{d}x\mathbf{M} + \mathbf{M}^{\mathrm{T}}\int_{x_A}^{x_B} \boldsymbol{\psi}\boldsymbol{\phi}^{\mathrm{T}}\mathrm{d}x\mathbf{w} + \mathbf{M}^{\mathrm{T}}\int_{x_A}^{x_B} \boldsymbol{\psi}\boldsymbol{\psi}^{\mathrm{T}}\mathrm{d}x\mathbf{M}\Bigg\} \\ & +\frac{k_1}{2}\Bigg\{\mathbf{w}^{\mathrm{T}}\int_{x_A}^{x_B} H\boldsymbol{\phi}_{,x}\boldsymbol{\phi}_{,x}^{\mathrm{T}}\,\mathrm{d}x\mathbf{w} + \mathbf{w}^{\mathrm{T}}\int_{x_A}^{x_B} H\boldsymbol{\phi}_{,x}\boldsymbol{\psi}_{,x}^{\mathrm{T}}\,\mathrm{d}x\mathbf{M} + \mathbf{M}^{\mathrm{T}}\int_{x_A}^{x_B} H\boldsymbol{\psi}_{,x}\boldsymbol{\phi}_{,x}^{\mathrm{T}}\,\mathrm{d}x\mathbf{w} + \mathbf{M}^{\mathrm{T}}\int_{x_A}^{x_B} H\boldsymbol{\psi}_{,x}\boldsymbol{\psi}_{,x}^{\mathrm{T}}\,\mathrm{d}x\mathbf{M}\Bigg\} \\ & +\frac{k}{2}\Bigg\{\mathbf{w}^{\mathrm{T}}\int_{x_A}^{x_B} \frac{\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}}{H}\,\mathrm{d}x\mathbf{w} + \mathbf{w}^{\mathrm{T}}\int_{x_A}^{x_B} \frac{\boldsymbol{\phi}\boldsymbol{\psi}^{\mathrm{T}}}{H}\,\mathrm{d}x\mathbf{M} + \mathbf{M}^{\mathrm{T}}\int_{x_A}^{x_B} \frac{\boldsymbol{\psi}\boldsymbol{\phi}^{\mathrm{T}}}{H}\,\mathrm{d}x\mathbf{w} + \mathbf{M}^{\mathrm{T}}\int_{x_A}^{x_B} \frac{\boldsymbol{\psi}\boldsymbol{\psi}^{\mathrm{T}}}{H}\,\mathrm{d}x\mathbf{M}\Bigg\} \\ & -\frac{1}{2\mathrm{EI}}\mathbf{M}^{\mathrm{T}}\int_{x_A}^{x_B} \boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\,\mathrm{d}x\,\mathbf{M} - \mathbf{M}^{\mathrm{T}}\boldsymbol{\theta} - \mathbf{w}^{\mathrm{T}}\mathbf{Q}\Bigg\} = 0, \end{aligned} \tag{35}$$

where $\mathbf{Q}$, $\boldsymbol{\theta}$ and $\mathbf{H}$ represent the vectors that their components are externally applied end forces, rotations and depth of the foundation layer at the ends, respectively. They can be written as follows:

$$\mathbf{Q}^{\mathrm{T}} = \{P_A, P_B\}, \quad \boldsymbol{\theta}^{\mathrm{T}} = \{-\theta_A, \theta_B\}, \quad \mathbf{H}^{\mathrm{T}} = \{H_A, H_B\}. \tag{36}$$

The end conditions of the beam resting on a Vlasov foundation come out to be mixed type conditions those can be given as $P_A = -S_A = -k_A\, w_A$ and $P_B = S_B = -k_B w_B$ for the first and the last beam elements of the straight beam, respectively.

Variations with respect to $\mathbf{w}$ and $\mathbf{M}$ are

$$\begin{aligned} \delta\mathbf{w}^{\mathrm{T}} \to & \int_{x_A}^{x_B} \boldsymbol{\phi}_{,x}\boldsymbol{\phi}_{,x}^{\mathrm{T}}\,\mathrm{d}x\mathbf{M} - P\Bigg\{\int_{x_A}^{x_B} \boldsymbol{\phi}_{,x}\boldsymbol{\phi}_{,x}^{\mathrm{T}}\,\mathrm{d}x\mathbf{w} + \int_{x_A}^{x_B} \boldsymbol{\phi}_{,x}\boldsymbol{\psi}_{,x}^{\mathrm{T}}\,\mathrm{d}x\mathbf{M}\Bigg\} \\ & - m\omega^2\Bigg\{\int_{x_A}^{x_B} \boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\,\mathrm{d}x\mathbf{w} + \int_{x_A}^{x_B} \boldsymbol{\phi}\boldsymbol{\psi}^{\mathrm{T}}\,\mathrm{d}x\,\mathbf{M}\Bigg\} \\ & + k_1\Bigg\{\int_{x_A}^{x_B} H\boldsymbol{\phi}_{,x}\boldsymbol{\phi}_{,x}^{\mathrm{T}}\,\mathrm{d}x\mathbf{w} + \int_{x_A}^{x_B} H\boldsymbol{\phi}_{,x}\boldsymbol{\psi}_{,x}^{\mathrm{T}}\,\mathrm{d}x\mathbf{M}\Bigg\} \\ & + k\Bigg\{\int_{x_A}^{x_B} \frac{\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}}{H}\,\mathrm{d}x\mathbf{w} + \int_{x_A}^{x_B} \frac{\boldsymbol{\phi}\boldsymbol{\psi}^{\mathrm{T}}}{H}\,\mathrm{d}x\mathbf{M}\Bigg\} - \mathbf{Q} = \mathbf{0}, \end{aligned} \tag{37}$$

$$\delta \mathbf{M}^{\mathrm{T}} \rightarrow \int_{x_A}^{x_B} \boldsymbol{\phi}_{,x}^{\mathrm{T}} \boldsymbol{\phi}_{,x}\, \mathrm{d}x \mathbf{w} + 2\int_{x_A}^{x_B} \boldsymbol{\psi}_{,x} \boldsymbol{\phi}_{,x}^{\mathrm{T}} \mathrm{d}x \mathbf{M} - P\left\{\int_{x_A}^{x_B} \boldsymbol{\psi}_{,x} \boldsymbol{\phi}_{,x}^{\mathrm{T}}\, \mathrm{d}x \mathbf{w} + \int_{x_A}^{x_B} \boldsymbol{\psi}_{,x} \boldsymbol{\psi}_{,x}^{\mathrm{T}}\, \mathrm{d}x \mathbf{M}\right\}$$
$$- m\omega^2 \left\{\int_{x_A}^{x_B} \boldsymbol{\psi}\boldsymbol{\phi}^{\mathrm{T}}\, \mathrm{d}x \mathbf{w} + \int_{x_A}^{x_B} \boldsymbol{\psi}\boldsymbol{\psi}^{\mathrm{T}}\, \mathrm{d}x \mathbf{M}\right\} + k_1\left\{\int_{x_A}^{x_B} H \boldsymbol{\psi}_{,x} \boldsymbol{\phi}_{,x}^{\mathrm{T}}\, \mathrm{d}x \mathbf{w} + \int_{x_A}^{x_B} H \boldsymbol{\psi}_{,x} \boldsymbol{\psi}_{,x}^{\mathrm{T}}\, \mathrm{d}x \mathbf{M}\right\}$$
$$+ k\left\{\int_{x_A}^{x_B} \frac{\boldsymbol{\psi}\boldsymbol{\phi}^{\mathrm{T}}}{H}\, \mathrm{d}x \mathbf{w} + \int_{x_A}^{x_B} \frac{\boldsymbol{\psi}\boldsymbol{\psi}^{\mathrm{T}}}{H}\, \mathrm{d}x \mathbf{M}\right\} - \frac{1}{\mathrm{EI}}\int_{x_A}^{x_B} \boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\, \mathrm{d}x \mathbf{M} - \boldsymbol{\theta} = \mathbf{0}. \tag{38}$$

The equations, representing the continuity of the nodal rotations and the equilibrium of the shear forces, can be written in matrix form by substituting Eq. (34) into Eqs. (37) and (38):

$$(\hat{\mathbf{K}} - P\hat{\mathbf{S}} - m\omega^2 \hat{\mathbf{M}})\mathbf{u} = \mathbf{f}, \tag{39}$$

where

$$\hat{\mathbf{K}} = \begin{bmatrix} k_1\mathbf{K}_1 + k\mathbf{K}_2 & \mathbf{K}_3 + k_1\mathbf{K}_4 + k\mathbf{K}_5 \\ \text{sym} & 2\mathbf{K}_6 + k_1\mathbf{K}_7 + k\mathbf{K}_8 - \mathbf{K}_9/\mathrm{EI} \end{bmatrix}, \quad \hat{\mathbf{S}} = \begin{bmatrix} \mathbf{K}_3 & \mathbf{K}_6 \\ \text{sym} & \mathbf{K}_{11} \end{bmatrix},$$
$$\hat{\mathbf{M}} = \begin{bmatrix} \mathbf{K}_9 & \mathbf{K}_{10} \\ \text{sym} & \mathbf{K}_{12} \end{bmatrix}, \quad \mathbf{u} = \begin{Bmatrix} \mathbf{w} \\ \mathbf{M} \end{Bmatrix}, \quad \mathbf{f} = \begin{Bmatrix} \mathbf{Q} \\ \boldsymbol{\theta} \end{Bmatrix}. \tag{40}$$

The implicit forms of sub-matrices in Eq. (40) are defined to be

$$\mathbf{K}_1 = \int_{x_A}^{x_B} H\boldsymbol{\phi}_{,x}\boldsymbol{\phi}_{,x}^{\mathrm{T}}\, \mathrm{d}x \quad \mathbf{K}_2 = \int_{x_A}^{x_B} \frac{\boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}}{H}\, \mathrm{d}x \quad \mathbf{K}_3 = \int_{x_A}^{x_B} \boldsymbol{\phi}_{,x}\boldsymbol{\phi}_{,x}^{\mathrm{T}}\, \mathrm{d}x$$
$$\mathbf{K}_4 = \int_{x_A}^{x_B} H\boldsymbol{\phi}_{,x}\boldsymbol{\psi}_{,x}^{\mathrm{T}}\, \mathrm{d}x \quad \mathbf{K}_5 = \int_{x_A}^{x_B} \frac{\boldsymbol{\phi}\boldsymbol{\psi}^{\mathrm{T}}}{H}\, \mathrm{d}x \quad \mathbf{K}_6 = \int_{x_A}^{x_B} \boldsymbol{\phi}_{,x}\boldsymbol{\psi}_{,x}^{\mathrm{T}}\, \mathrm{d}x$$
$$\mathbf{K}_7 = \int_{x_A}^{x_B} H\boldsymbol{\psi}_{,x}\boldsymbol{\psi}_{,x}^{\mathrm{T}}\, \mathrm{d}x \quad \mathbf{K}_8 = \int_{x_A}^{x_B} \frac{\boldsymbol{\psi}\boldsymbol{\psi}^{\mathrm{T}}}{H}\, \mathrm{d}x \quad \mathbf{K}_9 = \int_{x_A}^{x_B} \boldsymbol{\phi}\boldsymbol{\phi}^{\mathrm{T}}\, \mathrm{d}x$$
$$\mathbf{K}_{10} = \int_{x_A}^{x_B} \boldsymbol{\phi}\boldsymbol{\psi}^{\mathrm{T}}\, \mathrm{d}x \quad \mathbf{K}_{11} = \int_{x_A}^{x_B} \boldsymbol{\psi}_{,x}\boldsymbol{\psi}_{,x}^{\mathrm{T}} \mathrm{d}x \quad \mathbf{K}_{12} = \int_{x_A}^{x_B} \boldsymbol{\psi}\boldsymbol{\psi}^{\mathrm{T}}\, \mathrm{d}x \tag{41}$$

Since the basic system is a simple beam and C^0 continuity is sufficient for the functions of displacement and moment, the shape functions for beam element can be chosen as follows:

$$\boldsymbol{\phi}^{\mathrm{T}} = \{\varphi_A, \varphi_B\}, \quad \varphi_A = 1 - \xi, \quad \varphi_B = \xi = x/L,$$
$$\boldsymbol{\psi}^{\mathrm{T}} = \{\psi_A, \psi_B\}, \quad \psi_A = \frac{L^2}{6\mathrm{EI}}(2-\xi)(1-\xi)\xi, \quad \psi_B = \frac{L^2}{6\mathrm{EI}}(1+\xi)(1-\xi)\xi. \tag{42}$$

Substituting these expressions into Eqs. (41), the explicit forms of sub-matrices can be obtained as follows:

$$\mathbf{K}_1 = \frac{H_A}{L}(1-\eta)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$
$$\mathbf{K}_2 = \frac{L}{A}\left[\frac{1}{2\eta^2}\begin{bmatrix} 3\eta - 2 & 2-\eta \\ 2-\eta & -2-\eta \end{bmatrix} + \frac{1}{\eta^3}\begin{bmatrix} (\eta-1)^2 & \eta - 1 \\ \eta - 1 & 1 \end{bmatrix} \mathrm{Log}\left(\frac{1}{1-\eta}\right)\right],$$
$$\mathbf{K}_3 = \frac{1}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{K}_4 = \frac{H_A L}{24\mathrm{EI}}\,\eta\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix},$$
$$\mathbf{K}_5 = \frac{L^3}{H_A \mathrm{EI}}\left[\frac{1}{72\eta^4}\begin{bmatrix} 12 - 42\eta + 40\eta^2 - 7\eta^3 & -12 + 6\eta + 14\eta^2 - 5\eta^3 \\ -12 + 30\eta - 10\eta^2 - 3\eta^3 & 12 + 6\eta - 8\eta^2 - 3\eta^3 \end{bmatrix}\right.$$

$$+\frac{\eta-1}{6\eta^5}\begin{bmatrix}(\eta-1)(2\eta-1) & (\eta-1)(\eta+1)\\ (2\eta-1) & \eta+1\end{bmatrix}\mathrm{Log}\left(\frac{1}{1-\eta}\right)\Bigg]$$

$$\mathbf{K}_6=\mathbf{0}\quad \mathbf{K}_7=\frac{H_A L^3}{720(\mathrm{EI})^2}\begin{bmatrix}16-6\eta & 14-7\eta\\ 14-7\eta & 16-10\eta\end{bmatrix}$$

$$\mathbf{K}_8=\frac{L^5}{H_A\mathrm{EI}^2}\left[-\frac{1}{2160\eta^6}\begin{bmatrix}60-330\eta+620\eta^2-435\eta^3+62\eta^4+13\eta^5 \\ -60+150\eta+10\eta^2-165\eta^3+43\eta^4+11\eta^5\end{bmatrix}\right.$$

$$\times\begin{bmatrix}-60+150\eta+10\eta^2-165\eta^3+43\eta^4+11\eta^5\\ 60+30\eta-100\eta^2-45\eta^3+32\eta^4+10\eta^5\end{bmatrix}$$

$$\left.+\frac{(\eta-1)^2}{36\eta^7}\begin{bmatrix}(2\eta-1)^2 & (\eta+1)(2\eta-1)\\ (\eta+1)(2\eta-1) & (\eta+1)^2\end{bmatrix}\mathrm{Log}\left(\frac{1}{1-\eta}\right)\right]$$

$$\mathbf{K}_9=\frac{L}{6}\begin{bmatrix}2&1\\1&2\end{bmatrix}\quad \mathbf{K}_{10}=\frac{L^3}{360\mathrm{EI}}\begin{bmatrix}8&7\\7&8\end{bmatrix}\quad \mathbf{K}_{11}=\frac{1}{\mathrm{EI}}\mathbf{K}_{10}$$

$$\mathbf{K}_{12}=\frac{L^5}{15120(\mathrm{EI})^2}\begin{bmatrix}32&31\\31&32\end{bmatrix}\quad \eta=\frac{\alpha L}{H_A} \tag{43}$$

If the rigidity EI approaches to infinity, the matrices, with subscripts 4–8 and 10–12, disappear.

When the surface of rigid bed is near to the horizontal plane, α and of course η are close to zero. In that case, the elements of the matrices, with subscripts 2, 5 and 8, are seen numerically unstable. To achieve this difficulty, these matrices can be expanded into the power series as follows:

$$\mathbf{K}_2=\frac{L}{H_A}\left[\frac{1}{6}\begin{bmatrix}2&1\\1&2\end{bmatrix}+\sum_{k=1}^{m}\eta^k\begin{bmatrix}\frac{1}{k+3}-\frac{2}{k+2}+\frac{1}{k+1} & -\frac{1}{k+3}+\frac{1}{k+2}\\ -\frac{1}{k+3}+\frac{1}{k+2} & \frac{1}{k+3}\end{bmatrix}\right],$$

$$\mathbf{K}_5=\frac{L^3}{6H_A\mathrm{EI}}\left[\frac{1}{60}\begin{bmatrix}8&7\\7&8\end{bmatrix}\right.$$

$$\left.+\sum_{k=1}^{m}\eta^k\begin{bmatrix}-\frac{1}{k+5}+\frac{4}{k+4}-\frac{5}{k+3}+\frac{2}{k+2} & \frac{1}{k+5}-\frac{1}{k+4}-\frac{1}{k+3}+\frac{1}{k+2}\\ \frac{1}{k+5}-\frac{3}{k+4}+\frac{2}{k+3} & -\frac{1}{k+5}+\frac{1}{k+3}\end{bmatrix}\right]. \tag{44}$$

$$\mathbf{K}_8=\frac{L^5}{36H_A\mathrm{EI}^2}\left[\frac{1}{420}\begin{bmatrix}32&31\\31&32\end{bmatrix}\right.$$

$$\left.+\sum_{k=1}^{m}\eta^k\begin{bmatrix}\frac{1}{k+7}-\frac{6}{k+6}+\frac{13}{k+5}-\frac{12}{k+4}+\frac{4}{k+3} & -\frac{1}{k+7}+\frac{3}{k+6}-\frac{1}{k+5}-\frac{3}{k+4}+\frac{2}{k+3}\\ -\frac{1}{k+7}+\frac{3}{k+6}-\frac{1}{k+5}-\frac{3}{k+4}+\frac{2}{k+3} & \frac{1}{k+7}-\frac{2}{k+5}+\frac{1}{k+3}\end{bmatrix}\right],$$

where the first terms correspond to the case $\eta=0$.

6. Examples

The elastic properties of a straight beam on a two-parameter Vlasov foundation can be characterized by a single parameter as follows [15]:

$$\beta L = \sqrt[3]{\frac{E_0 b L^3}{16(1-v_0^2)\mathrm{EI}}}, \tag{45}$$

where L is the length of the beam. Vlasov has classified this kind of beams in three categories for the static problems as:

(a) If $\beta L < 0.86$, beam is called as a *short beam* that behaves as a rigid member on an elastic foundation under any kind of loading, i.e., the curvature is negligible throughout the beam.
(b) If $0.86 < \beta L < 1.85$, beam is called as an *intermediate beam*. When a concentrated load applied to one of the ends, any quantity observed at the other end and the curvature throughout the beam are not negligible.
(c) If $\beta L > 1.85$, beam is called as a *long beam*. When a concentrated load applied to one of the ends, any quantity observed at the other end is negligible.

The uncertain intervals of βL have been given by Vlasov by two assumptions, [15]: the depth of foundation layer is infinite and the transverse displacements of foundation changes exponentially as $\phi_d = e^{-\gamma y}$ with $\gamma = 3/2$.

In the sample problems, beams have been divided into identical beam elements. P^* and ω^* are the fundamental buckling load and the fundamental free vibration frequency parameter, respectively. The thickness of the foundation and the width of cross-section are selected to be equal to b. L is the length of beam. Unless other numerical values are specified, the parameters are selected as $b = 1\,\mathrm{m}$, $\mathrm{EI} = 3\times10^5\,\mathrm{kN\,m^2}$, $L = 10\,\mathrm{m}$, $E_0 = 10^5\,\mathrm{kN/m^2}$ and $v_0 = 0.25$.

The sub-matrices expanded into the power series as in Eq. (44) are calculated by taking the first 20 terms, so that the relative error on any element of a matrix is limited to $\pm10^{-8}$ for $0 \leqslant \eta \leqslant 2/5$. The relative error is of course zero, when $\eta = 0$.

6.1. Sensitivity to number of elements

The free vibration and the static stability analyses of the system in Fig. 4 are performed to show the variation of results with respect to the number of beam elements n. The beam, with the parameters given above, can be classified as a long beam, $\beta L = 0.61 < 0.85$. the other parameters are selected as $H_0 = 5\,\mathrm{m}$ and $\alpha = 0.4$. Then, the coefficients of the elastic springs are obtained as $k_A = 35421.0$ and $k_B = 40754.3\,\mathrm{kN/m}$.

The smallest five of static buckling loads and free vibration parameters are determined by using different number of beam elements. The results are illustrated in Fig. 5. It is easily seen that the approximation rate of the method is reasonable.

The first four of mode shapes and corresponding moment diagrams, which are obtained by the analyses of static buckling and free vibration, are illustrated in Figs. 6 and 7, respectively.

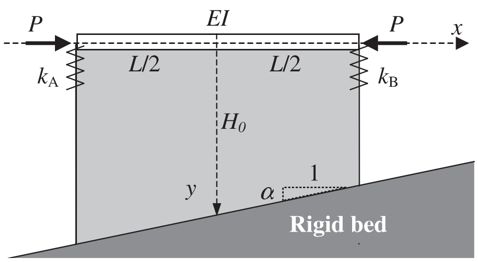

Fig. 4. Beam and wedge foundation with fictitious springs for sensitivity tests.

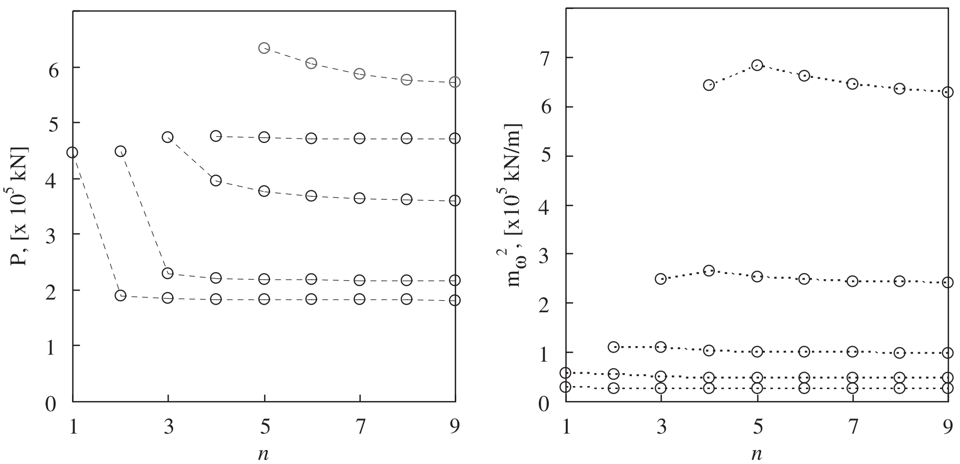

Fig. 5. Variation of buckling load P and free vibration parameter $m\omega^2$ with respect to number of elements n. $H_0 = 5\,\text{m}$ and $\alpha = 0.4$.

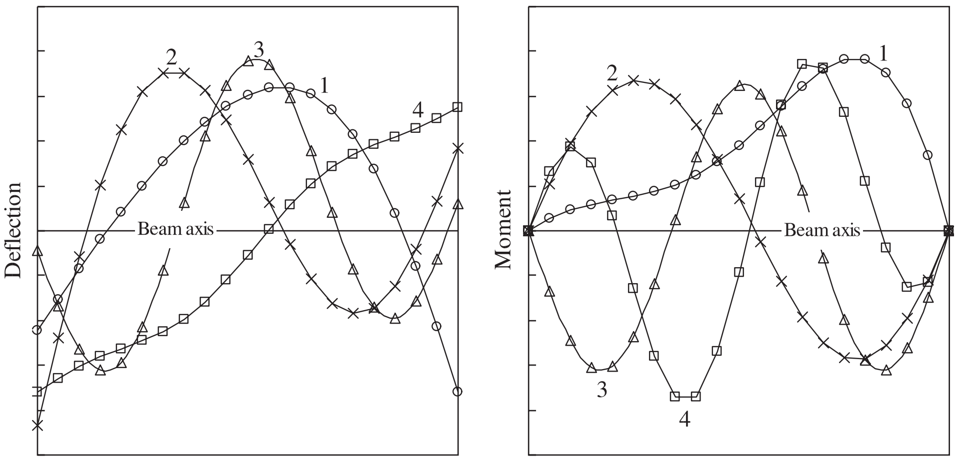

Fig. 6. Mode shapes of buckling and corresponding moment diagrams. $H_0 = 5\,\text{m}$, $\alpha = 0.4$ and $n = 14$.

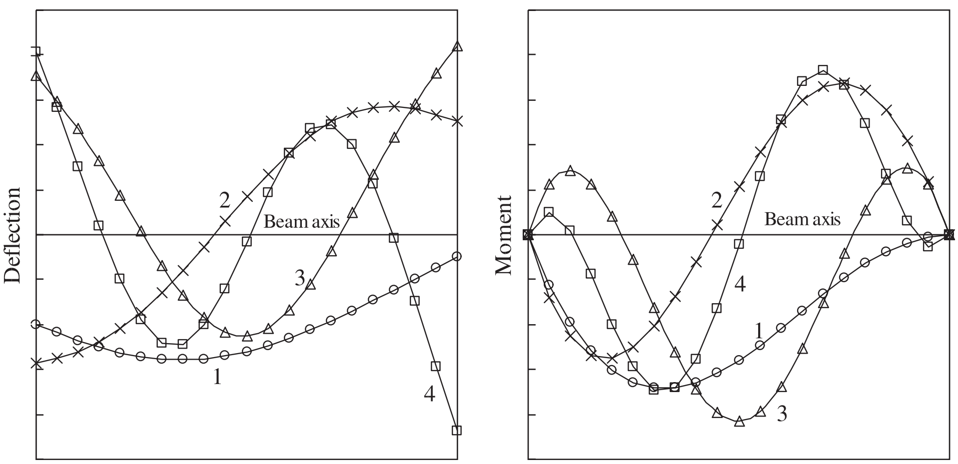

Fig. 7. Mode shapes of free vibration and corresponding moment diagrams. $H_0 = 5\,\text{m}$, $\alpha = 0.4$ and $n = 14$.

6.2. Sensitivity to bending rigidity, slope and depth

The free vibration and the static stability analyses of the system in Fig. 4. are also performed to show how the system is sensitive to the bending rigidity of the beam, slope of the rigid bed and the average depth of the foundation layer.

The effects of the bending rigidity on the fundamental buckling load and natural frequency parameter are illustrated in Fig. 8 in which the uncertain intervals of the short and long beam categories are seen.

The first three of static buckling loads and natural frequency parameters, except the fundamental frequency parameter, increase with the slope of the rigid bed as illustrated in Fig. 9.

The effects of average depth of foundation layer on the first three of the buckling loads and the natural frequency parameters are illustrated in Fig. 10. Unexpectedly, these quantities have the minimum values for the specific values of the depth of the rigid bed. Only fundamental frequency parameter always decreases with the increasing depth.

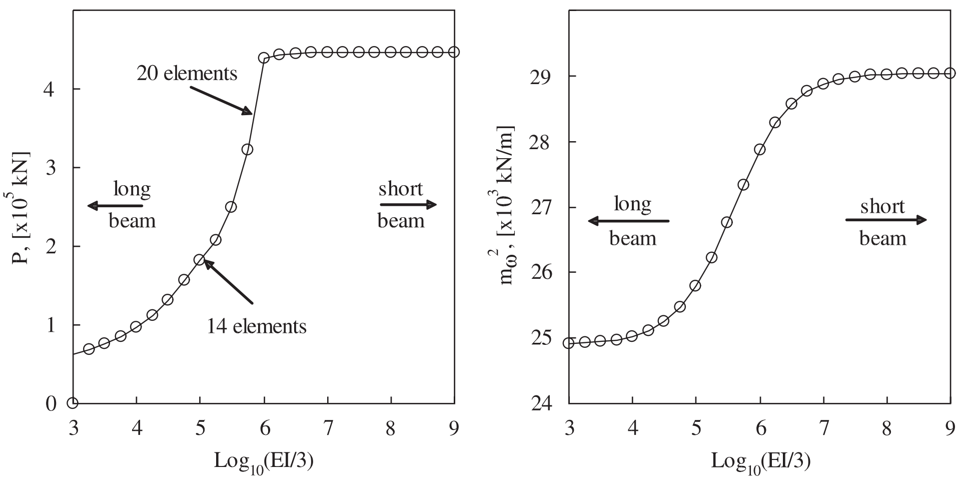

Fig. 8. Variation of parameters of fundamental buckling load and natural frequency with respect to bending rigidity EI (kN m^2). $H_0 = 5$ m and $\alpha = 0.4$.

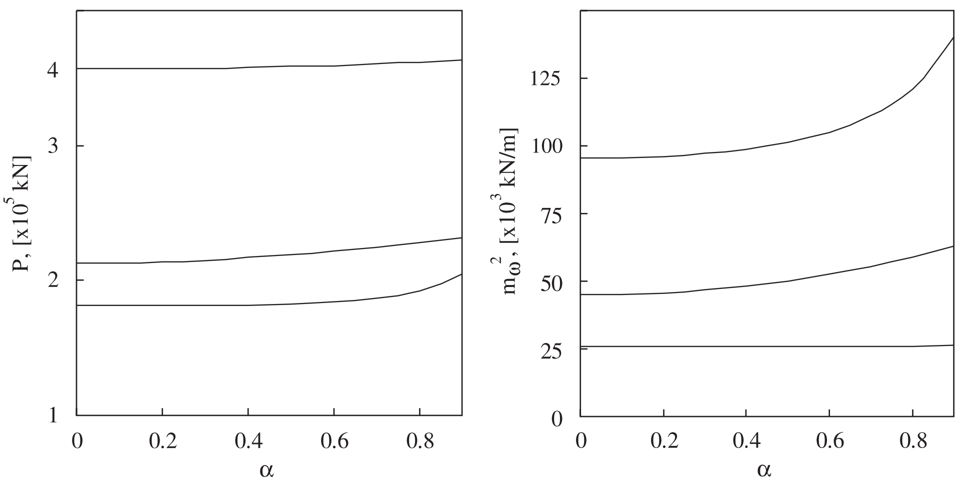

Fig. 9. Variation of the smallest three of buckling loads and natural frequency parameters with respect to slope of rigid bed α. $H_0 = 5$ m and $n = 20$.

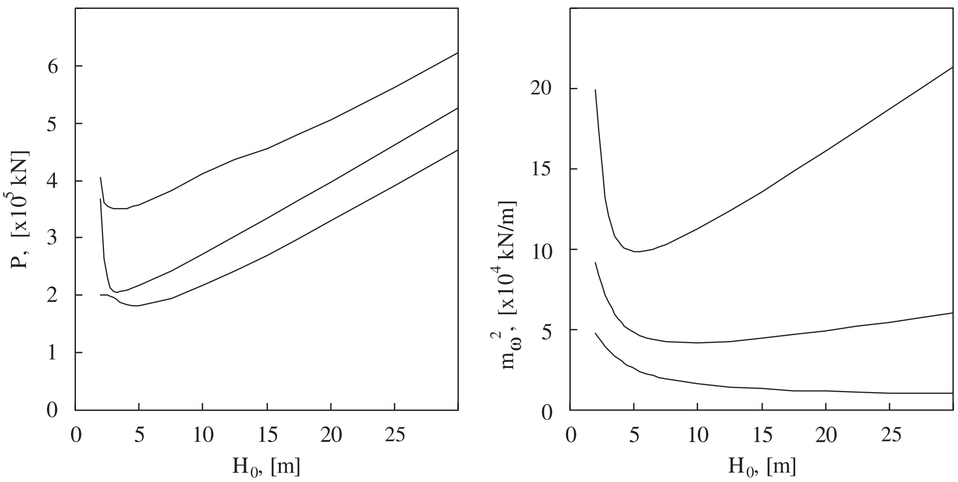

Fig. 10. Variation of the smallest three of buckling loads and natural frequency parameters with respect to average depth of rigid bed H_0. $\alpha = 0.4$ and $n = 20$.

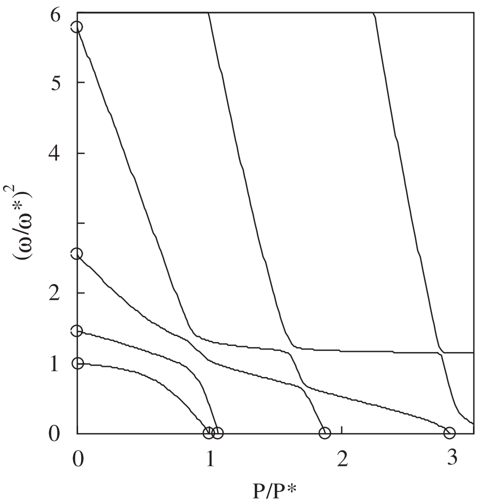

Fig. 11. Dimensionless natural frequency parameters of pre-stressed beam on elastic wedge foundation. Circled marks represent pure cases. Mode numbers are depicted on curves. $H_0 = 3\,\text{m}$, $\alpha = 0.1$ and $n = 40$.

6.3. Free vibration of pre-stressed beam

In this example, free vibration problem of a pre-stressed beam on the elastic foundation, given in Fig. 4, is investigated. Here, the parameters are taken as $\alpha = 0.1$ and $H_0 = 3\,\text{m}$. The coefficients of the elastic springs, fundamental buckling loads and fundamental vibration frequency parameters are obtained as $k_A = 37069.3$, $k_B = 38402.6\,\text{kN/m}$, $P^* = 184851.4\,\text{kN}$ and $m\omega^{*2} = 39120.8\,\text{kN/m}$, respectively.

The relationship between natural frequency parameters and the axial static load is illustrated in Fig. 11. Circles in this figure show the pure buckling loads and the pure free vibration frequency parameters.

7. Conclusions

Eigenvalue problems of the Euler–Bernoulli beams on a Vlasov foundation in the shape of a wedge have been investigated by employing the mixed finite element method. Effects of the end moments have been taken into consideration in the interpolation function for displacement.

The displacement and the moment have been selected as primary variables, while their first derivatives have been denoted as secondary variables. The behavior matrices of the beam element have been obtained in mixed form using the weak formulation satisfying equilibrium and compatibility equations.

Although it is important for practical applications, the inertia of the foundation layer has been disregarded for the sake of simplicity.

The approximation rate of the presented method is seemed as reasonable for the practical purposes.

In the literature, the beams, on a Vlasov-type elastic layer with the horizontal rigid bed, have been classified into three categories such as long, intermediate and short beam. It is demonstrated in this study that the beams on a wedge foundation layer can also be classified in the same way.

The parameters of buckling loads and natural frequencies have the minimum values for the specific values of the depth of the rigid bed. Only the fundamental frequency parameter decreases with the increasing depth. This fact can be explained as a result of using a linear shape function for the deep foundation layer of the Vlasov type. In fact, the use of a linear shape function is not suitable for the deep foundation layer. So, the presented formulation is accurate for shallow layers only.

It is shown from the free vibration analysis of the pre-stressed beam that the frequency parameter decreases with the increasing axial load.

Uplift problems, the dynamic stability analysis of a beam on an elastic foundation layer, and consideration of the inertia of the foundation will be the subjects of future studies.

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